Distances in the Hyperbolic Plane and the Hyperbolic Pythagorean Theorem

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Terminology and notation. I'm going to use H_1-*distance* to mean the distance between two points of the upper half-plane as a model for hyperbolic geometry. Similarly, by H_2-*distance* I will mean the distance between two points of the unit disc as a model for hyperbolic geometry. I will denote the H_1-distance from a to b in the upper half-plane by $H_1(a, b)$ and the H_2-distance from a to b in the unit disc by $H_2(a, b)$.

Distances in the upper half-plane model. Suppose $a, b \in \mathbb{C}$ are any two points in the upper half-plane.

Consider first the case where a, b lie on a vertical line, say $a = x + iy_1$ and $b = x + iy_2$. This line is the image of the map $\gamma(t) = it$ for $t \in \mathbb{R}$. $|dz|/\operatorname{Im}(z)$ is the element of arc length in the upper half-plane model, so

$$H_1(a, b) = \left| \int_{y_1}^{y_2} \frac{dt}{t} \right| = |\log(y_2/y_1)|.$$

If a, b do not lie on a vertical line, it is necessary to determine a geodesic connecting them. Let c denote the intersection of the real axis and the perpendicular bisector of the segment joining a and b. Then a and b lie on a semicircle M centered at c and orthogonal to the real axis; let p denote one of the intersection points of this circle and the real axis. Suppose C is any circle centered at p, and let q be the intersection in the upper half-plane of C and M.

Apply reflection in C to the scene hence constructed, so M is sent to a vertical line. By composing suitable fractional linear transformations with reflection in the imaginary axis, one sees that reflection in C must be an orientation-reversing isometry of the upper half-plane when endowed with hyperbolic geometry, so the arc of M connecting a and b and its image under this reflection, a vertical line segment connecting the images of a and b, have the same H_1-length. Therefore, this arc is a geodesic connecting a and b, so if $\tilde{a}$ and $\tilde{b}$ are the images of a and b under this reflection then the previous result about lengths of vertical line segments yields

$$H_1(a, b) = \left| \log \left(\frac{\operatorname{Im}(\tilde{a})}{\operatorname{Im}(\tilde{b})} \right) \right|.$$

Another (more explicit) formula can be found after distances have been understood in the unit disc.

Distances in the unit disc model. Consider the fractional linear transformation S that sends $\infty \mapsto i$ and $\pm 1 \mapsto \pm 1$ (i.e., 1 and -1 are each fixed points). S sends the real axis to the boundary of the unit disc and, since fractional linear transformations preserve the orientation of circles, it sends the upper half-plane to the disc's interior.

With this conformal map in hand, the H_2-distance between two points a, b in the unit disc is the H_1-distance between their preimages $S^{-1}(a), S^{-1}(b)$ in the upper half-plane, and in this way the unit disc inherits a metric from the metric of the upper half-plane.

Write D^1 for the (interior of) the unit disc and suppose $\gamma : [0,1] \to D^1$ is a piecewise continuously differentiable curve. If $H_i(\gamma)$ denotes the length of the curve γ in the appropriate model, then the previous paragraph simply says that $H_2(\gamma) = H_1(S^{-1} \circ \gamma)$. Writing $T = S^{-1}$, we have

$$H_1(T \circ \gamma) = \int_{T \circ \gamma} \frac{1}{\mathrm{Im}(z)}\, |dz| = \int_0^1 \frac{1}{\mathrm{Im}((T \circ \gamma)(t))} |(T \circ \gamma)'(t)|\, dt$$

$$= \int_0^1 \frac{1}{\mathrm{Im}(T(\gamma(t)))} |T'(\gamma(t))|\, |\gamma'(t)|\, dt$$

$$= \int_\gamma \frac{1}{\mathrm{Im}(T(z))} |T'(z)|\, |dz|.$$

Now T has the form $T(z) = (iz - 1)/(-z + i)$. Thus

$$\mathrm{Im}(T(z)) = \mathrm{Im}\left(\frac{(iz - 1)(-\overline{z} - i)}{|-z + i|^2} \right) = \frac{1 - |z|^2}{|-z + i|^2}$$

and

$$|T'(z)| = \left| \frac{-2}{(-z + i)^2} \right| = \frac{2}{|-z + i|^2},$$

so we obtain

$$H_2(\gamma) = \int_\gamma \frac{2}{1 - |z|^2}\, |dz|.$$

This provides the general formula for computing distances in the Poincaré disc and also shows that $2/(1 - |z|^2)\, dz$ is the element of arc length in this model.

Now any diameter of the unit disc is a geodesic, so if z is a point in the unit disc, then the Euclidean segment from 0 to z is also a hyperbolic segment from 0 to z. We have hence

$$H_2(0, z) = \int_0^{|z|} \frac{2}{1 - t^2}\, dt = 2 \tanh^{-1}(|z|) = \log\left(\frac{1 + |z|}{1 - |z|} \right).$$

The orientation-preserving isometries of the Poincaré disc are precisely the fractional linear transformations preserving the unit disc. Let us identify this group of motions. If M is a fractional linear transformation preserving the unit disc, then the following data will determine M: (1) the preimage $M^{-1}(0)$ of the origin; (2) the image $M(1)$ of 1 on the disc's boundary. Indeed, $M^{-1}(0)$ and $1/\overline{M^{-1}(0)}$ are symmetric with respect to the unit circle and their images must therefore also be symmetric with respect to the unit circle, so $M(1/\overline{M^{-1}(0)}) = \infty$. Hence, M has the form

$$z \mapsto \alpha \frac{z - M^{-1}(0)}{\overline{M^{-1}(0)} z - 1}$$

for some complex number $\alpha \in \mathbb{C}$. But

$$M(1) = \alpha \frac{1 - M^{-1}(0)}{\overline{M^{-1}(0)} - 1},$$

so

$$\alpha = \frac{M(1)(\overline{M^{-1}(0)} - 1)}{1 - M^{-1}(0)}.$$

Moreover, $|\alpha| = |M(1)| \cdot |(\overline{M^{-1}(0)} - 1)/(1 - M^{-1}(0))| = 1 \cdot 1$ since $M(1)$ lies on the unit circle, so $\alpha = e^{i\theta}$ for some $\theta \in \mathbb{R}$. In short, the fractional linear transformations preserving the unit disc have the form

$$z \mapsto e^{i\theta} \frac{z - \beta}{\overline{\beta} z - 1}$$

where β is the preimage of 0 under this map. Denote this map by $M(\theta, \beta)$.

This not only specifies the motions of the Poincaré disc, but it also furnishes an explicit one-parameter family of fractional linear transformations sending an arbitrary point β in the disc to the origin. If $\theta \in \mathbb{R}$ is arbitrary and a, b are points of the unit disc, then it follows that

$$H_2(a, b) = H_2(M(\theta, a)(a), M(\theta, a)(b)) = H_2\left(0, e^{i\theta}\frac{b-a}{\overline{a}b-1}\right).$$

Choose θ such that $e^{i\theta}\frac{b-a}{\overline{a}b-1} = |\frac{b-a}{\overline{a}b-1}|$; then this becomes

$$H_2(a, b) = H_2\left(0, \left|\frac{b-a}{\overline{a}b-1}\right|\right) = 2\tanh^{-1}\left|\frac{b-a}{\overline{a}b-1}\right|.$$

Distances in the upper half-plane model, cont'd. Recalling that S is the map from the upper half-plane to the unit disc, the definitions have been set up so that if a, b are points of the upper half-plane, then $H_1(a, b) = H_2(S(a), S(b))$. Therefore,

$$H_1(a, b) = 2\tanh^{-1}\left|\frac{\dfrac{ib+1}{b+i} - \dfrac{ia+1}{a+i}}{\dfrac{-i\overline{a}+1}{\overline{a}-i}\dfrac{ib+1}{b+i} - 1}\right| = 2\tanh^{-1}\left|\frac{a-b}{a-\overline{b}}\right|.$$

Since $2\tanh^{-1}(x) = \log((1+x)/(1-x))$ if $|x| < 1$, we obtain after clearing denominators the equivalent expression

$$H_1(a, b) = \log\left(\frac{|a-\overline{b}| + |a-b|}{|a-\overline{b}| - |a-b|}\right).$$

The Pythagorean theorem. I'll derive Pythagoras' theorem in the unit disc model. Suppose A, B, C are the vertices of a hyperbolic triangle in the unit disc. Write $a = |BC|$, $b = |AC|$, $c = |AB|$ for the Euclidean lengths of the Euclidean segments joining the vertices, $\tilde{a}, \tilde{b}, \tilde{c}$ for the H_2-lengths of the hyperbolic segments joining the vertices, and $\alpha = \angle BAC$, $\beta = \angle ABC$, $\gamma = \angle ACB$ for the measures of the angles (with respect to the hyperbolic edges joining the vertices, not the Euclidean edges).

Because fractional linear transformations preserving the unit disc are conformal and distance preserving maps of the Poincaré disc to itself, any right triangle $\triangle ABC$ (with α the right angle) can always be transformed into a triangle with $A = 0$, $B = c$, and $C = bi$ where the lengths of the sides and angles are unaffected; it suffices to prove the theorem for this particularly nice triangle. Since c is the Euclidean distance from the origin to B, the H_2-distance from the origin to B is simply $\tilde{c} = 2\tanh^{-1}(c)$.

Thus $c = \tanh(\tilde{c}/2)$, $b = \tanh(\tilde{b}/2)$, and

$$\tanh(\tilde{a}/2) = \left|\frac{C-B}{1-\overline{C}B}\right|.$$

It is an identity that

$$\cosh(\tilde{a}) = \frac{1+\tanh^2(\tilde{a}/2)}{1-\tanh^2(\tilde{a}/2)},$$

so we obtain in fact

$$\cosh(\tilde{a}) = \frac{|1-\overline{C}B|^2 + |C-B|^2}{|1-\overline{C}B|^2 - |C-B|^2} = \frac{(1+|B|^2)(1+|C|^2) - 2(\overline{B}C + B\overline{C})}{(1-|B|^2)(1-|C|^2)}$$

$$= \frac{1+|B|^2}{1-|B|^2}\cdot\frac{1+|C|^2}{1-|C|^2} - \frac{2(\overline{B}C + B\overline{C})}{(1-|B|^2)(1-|C|^2)}.$$

But everything has been constructed so that $\overline{B}C + B\overline{C} = cbi - cbi = 0$ and the right-most term drops out. The first fraction of the first term is

$$\frac{1 + |B|^2}{1 - |B|^2} = \frac{1 + \tanh^2(\tilde{b}/2)}{1 - \tanh^2(\tilde{b}/2)} = \cosh(\tilde{b})$$

and similarly the second fraction of the first term is $\cosh(\tilde{c})$. The theorem is proved.