# The Christoffel-Schwarz Formula

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Suppose $\Omega$ is a bounded simply connected region whose boundary is a simple polygon $P$. Suppose $z_1, z_2, \ldots, z_n$ are the consecutive vertices of $P$. Write $\alpha_k \pi$ for the interior angle at vertex $z_k$ and $\beta_k \pi = \pi - \alpha_k \pi$ for the exterior angle at the same vertex. Finally, suppose $f$ is a bijective conformal map sending $\Omega$, the interior of the polygon $P$, onto the open unit disc $\mathbb{D}$. We wish to find an explicit representation for the inverse $F$ of $f$, and to do so we will in fact find a representation for the derivative $F'$ and then find $F$ by integration. This is accomplished by a study of the behavior of $f$ near the corners of $P$.

We depart from the following two facts which we will not prove: (1) $f$ can be extended by continuity to the open line segment connecting two consecutive vertices of $P$ and, further, these open segments are mapped onto arcs of the unit circle; (2) we assume the reflection principle of Schwarz.

We assume $f$ has been extended as in (1) for the rest of this document. We begin by drawing a small circle around the vertex $z_k$. The intersection of the interior of this circle and $\Omega$ is a circular sector $S_k$. We introduce the new coordinate $\zeta = (z - z_k)^{1/\alpha_k}$. By suitably choosing the branch of this function the sector $S_k$ is mapped onto a half-disc $S'_k$ with $z_k$ sent the origin. We now "pull back" $f$ from $S_k$ to $S'_k$ and switch coordinates from $z$ to $\zeta$. Hence we consider the map $g(\zeta) = f(\zeta^{\alpha_k} + z_k)$ from $S_k$ onto some subset of $\mathbb{D}$.

Suppose $K$ is a compact set in $g(S_k) \subseteq \mathbb{D}$. Since $g$ is a homeomorphism from $S_k$ onto $g(S_k)$, the preimage $g^{-1}(K)$ is a compact subset of $S_k$. Suppose now that $\{\zeta_k\}_{k=1}^{\infty}$ is a sequence of points in $S_k$ tending to the diameter of $S_k$. For each point $a \in S_k$ pick a number $\epsilon_\zeta$ such that the disc $0 \leq |\zeta - a| < \epsilon_\zeta$ is contained entirely in $S_k$. The discs $0 \leq |\zeta - a| < \epsilon_\zeta$ for $\zeta \in S_k$ form an open cover of $S_k$. Therefore, $g^{-1}(K)$ is covered by finitely many of these discs, and so there is an integer $n_0$ such that $\zeta_n$ is not in $g^{-1}(K)$ for $n \geq n_0$. But then $g(\zeta_n)$ is not in $K$ for $n \geq n_0$. We hence find that the sequence $\{g(\zeta_k)\}_{k=1}^{\infty}$ eventually exits from every compact subset of $g(S_k)$. Since the diameter of $S_k$ is mapped onto an arc of the unit circle, this means that $|g(\zeta)| \to 1$ as $\zeta$ approaches the diameter of $S_k$.

The reflection principle of Schwarz provides an analytic continuation of $g$ to the whole disc. In fact, the analytic continuation of $g$ satisfies $g(\zeta^*) = (g(\zeta))^*$ where $z^*$ is the reflection of $z$ in the diameter of the disc.

We replace $g$ by this analytic continuation. The previous computation allows us to conclude by sending $\xi \to 0$ that $f(z)$ has a limit $w_k = e^{i\theta_k}$ as $z \to z_k$, and hence the arcs of the unit circle that are the images of the sides of the edges of the polygon meeting at $z_k$ have a point $w_k$ in common.

We now demonstrate that the arcs of the circle are otherwise disjoint. To do this, we draw a small rectangle inside $\Omega$ with one edge on the segment joining $z_k$ to $z_{k+1}$, and we denote the enclosed region along with the rectangular boundary $R$. Suppose $a, b$ are the vertices of $R$ on the segment $[z_k, z_{k+1}]$. The image of $R$ under $f$ is a region $U$ in $\mathbb{D}$ whose boundary contains the arc of the unit circle corresponding to the segment $[a, b]$. But $f$ is analytic and injective on $R$, so if the boundary of $R$ is given a positive orientation the boundary of $U$ will inherit this orientation. This implies that the arc of the circle corresponding to $[a, b]$ must have a positive orientation relative to the open disc $\mathbb{D}$. Since this is true for every edge of the polygon, it follows that the images of the edges are arcs of the circle that are disjoint except at the endpoints.

Thus $f$ maps the closure of $\Omega$ onto the closed unit disc, the points $z_k$ to the points $w_k$, and the edges of the polygon to arcs of the unit circle connecting the images of the respective vertices.

Since $g$ has been analytically continued to the full disc, it is analytic at the origin, so it has the convergent Taylor development

$$f(z_k + \zeta^{\alpha_k}) = w_k + \sum_{m=1}^{\infty} a_m \zeta^m.$$

Now suppose $g'(0) = 0$. Then by theorems on local correspondence $g$ must be at least two-to-one in some neighborhood of zero, i.e., there must be points $a, b$ nearby zero such that $g(a) = g(b)$. Let $T_k'$ denote the reflection of the half-disc $S_k'$ in its diameter. $g$ is one-to-one on $S_k'$ and hence also on its reflection $T_k'$ in the diameter, so it must be that $a \in S_k'$ and $b \in T_k'$. But by the reflection principle we find that $a$ will be contained in $\mathbb{D}$ and $b$ will be in the exterior of $\mathbb{D}$, a contradiction.

Therefore, the series that occurs in the right member of the above equation has an inverse. Writing $w = f(z_k + \zeta^{\alpha_k})$, we find

$$\zeta = \sum_{m=1}^{\infty} b_m (w - w_k)^m,$$

valid in some neighborhood of $w_k$. Moreover, $b_1 \neq 0$ since this series also is invertible.

We raise both sides of the most recent equation to the power $\alpha_k$ and obtain

$$F(w) - z_k = (w - w_k)^{\alpha_k} G_k(w),$$

where $G_k$ is analytic and nonzero near $w_k$. Differentiating both sides and then dividing by $(w - w_k)^{\alpha_k - 1}$ gives

$$F'(w)(w - w_k)^{\beta_k} = \alpha_k G_k(w) + (w - w_k) G_k'(w)$$

since $\beta_k = 1 - \alpha_k$. The limit of the right member as $w \to w_k$ is $\alpha_k G_k(w_k) \neq 0$, and hence the left member $F'(w)(w - w_k)^{\beta_k}$ represents a function that is analytic and nonzero at $w_k$. Therefore, the product

$$H(w) = F'(w) \prod_{k=1}^{n} (w - w_k)^{\beta_k}$$

is analytic and nonzero in the closed unit disc. We now show that $H(w)$ is a constant.

To accomplish this, suppose that $w = e^{i\theta}$ with $\theta_k < \theta < \theta_{k+1}$ for some $k$. Now $\arg F'(e^{i\theta})$ is the angle between the tangent to the unit circle at $e^{i\theta}$ and the tangent to the image of the unit circle at $F(e^{i\theta})$. But the latter tangent has constant argument on the segment joining $z_k$ to $z_{k+1}$ and the former tangent has argument $\theta + \pi/2$. Moreover, the argument of $w - w_k$ is $\theta/2$ plus a constant. We find thus that $\arg H(w)$ differs from

$$-\theta + \frac{\theta}{2} \sum_{k=1}^{n} \beta_k$$

by a constant independent of $\theta$. From geometric considerations we find that the sum is 2 and hence this quantity reduces to 0. Therefore, $\arg H(w)$ is constant on the arcs connecting consecutive points $w_k$ and $w_{k+1}$. By continuity it is constant on the whole circle. But the argument is the harmonic function $\operatorname{Im}\log(H(w))$, and harmonic functions are subject to the minimum and maximum principles. Using both principles at once shows that $\operatorname{Im}\log(H(w))$ is constant in the unit disc. An analytic function with constant imaginary part reduces to a constant, and hence $\log(H(w))$ and so also $H(w)$ are constant.

We finally obtain by integration

$$F(w) = C \int_0^w \prod_{k=1}^n (w - w_k)^{-\beta_k} \, dw + C'$$

where $C, C'$ are constants and the path of integration is arbitrary.