Euler's Formula and the Sylvester-Gallai Theorem

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1 Overview

The Sylvester-Gallai theorem is a surprising theorem of geometry that asserts that given three or more points in the plane, not all collinear, there must exist a line containing precisely two of the points. It will be our object to prove this theorem, and we will do so by developing a few intermediate but still fascinating results, the first of which is Euler's famous formula relating the vertices, edges, and faces of a connected planar graph.

2 Graph-Theoretic Preliminaries

We will be using the language of graph theory to discuss some elegant topological concepts, so we begin by introducing this language.

Definition 1. A **graph** is an ordered pair (V, E) where V and E are sets together with a map ϕ from E to $V \cup [V]^2$, where $[V]^2$ is the set of two-element subsets of V. Thus the map ϕ assigns to each edge either one or two vertices, called its **endpoints**. The elements of V are called **vertices** and the elements of E are called **edges**.

From now on will we allow ourselves to use more informal language regarding graphs for the sake of actually accomplishing something in a reasonable amount of time. We will, for instance, not worry about the map ϕ and simply speak of an edge connecting a vertex to itself or connecting two different vertices. This definition is meant to give the flavor of a completely formal approach to graph theory.

It is easy to imagine graphs which contain two vertices that aren't connected by any sequence of edges in the graph. The following definitions allow us to discuss such graphs precisely.

Definition 2. A **path** is a nonempty graph (V, E) such that $V = \{x_0, x_1, \ldots, x_k\}$ and the edge set E consists precisely of edges connecting x_0 to x_1, x_1 to x_2, $\ldots$, x_{k-1} to x_k.

It's convenient to refer to x_0 as the starting vertex or initial vertex of the path and x_k as the final, terminal, or ending vertex.

Definition 3. A graph is **connected** if every two distinct vertices of the graph occur as the initial and terminal vertices respectively of some path contained in the graph.

We can imagine representing a graph as a set of points in Euclidean space and realizing the edges as smooth arcs connecting pairs of (not necessarily distinct) vertices. We will say that a graph is **planar** if it can be drawn in such a way that its vertices are coplanar and the edges do not intersect except possibly at their endpoints. It will be important for us to observe that it is equivalent to require that the vertices and edges lie on the surface of a *sphere* instead of a plane. For instance, stereographic projection provides an explicit one-to-one correspondence between a plane and the surface of a sphere (minus one point).

By a **plane graph** we will mean a drawing of a planar graph in a plane (or on the surface of a sphere) such that its edges do not intersect except possibly at their endpoints. (So a planar graph is a graph that can be drawn in this way; a plane graph is one that is already drawn in this way.) We remark without proof that a finite plane graph divides the plane into finitely many connected regions, called **faces**. Note that a plane graph drawn in a plane will determine an unbounded face but the same plane graph drawn on a sphere will determine only bounded faces.

We record two final definitions:

Definition 4. A **loop** in a graph is an edge connecting a vertex to itself. A graph is **simple** if it contains no loops and any two distinct vertices are connected by at most one edge.

Definition 5. The **degree** of a vertex in a graph is the number of edges attached to it.

3 Euler's Formula

Theorem 6. *Suppose a drawing of a finite connected planar graph has $V \geq 1$ vertices, E edges, and F faces. Then*

$$V - E + F = 2.$$

This result is often called **Euler's formula**, although many other famous equations of mathematics also go by this monicker.

Proof. Any finite connected planar graph can be built from a single vertex by applying finitely many of the following four rules one at a time:

(1) Add a vertex to an existing edge.

(2) Add an edge from a vertex back to itself (a loop).

(3) Add an edge between two existing vertices.

(4) Add an edge and a vertex to an existing vertex.

Not every graph built from these operations will be planar, but the graphs that this theorem is concerned with can always be built with this procedure. We proceed by induction on the number of steps required to construct the graph from an initial vertex using these operations.

In the base case the graph is just a vertex and no steps are required. The formula holds since $V = 1$, $F = 1$, and $E = 0$.

Suppose now the result holds when k steps are needed to construct the graph. It suffices by the induction hypothesis to show that the quantity $V - E + F$ will not be changed by any of the operations $(1) - (4)$, for any graph that requires $k + 1$ steps is obtained from a graph requiring k steps by performing an extra step. We record how each operation affects the quantities $V, E, F, V - E + F$ in the following table:

Operation	ΔV	ΔE	ΔF	$\Delta(V - E + F)$
(1)	$+1$	$+1$	0	0
(2)	0	$+1$	$+1$	0
(3)	0	$+1$	$+1$	0
(4)	$+1$	$+1$	0	0

It is perhaps unclear why operation (3) should increase the number of faces by 1. In fact, this is because the graph to which operation (3) is applied is assumed to be *connected.* The two vertices that are joined by an edge by operation (3) are hence already joined by a path, and this path along with the new edge together will constitute the boundary of a region of the surface the graph resides in (say a plane or the sphere S^2). It may be that this is not a maximal connected region, for there may be multiple paths connecting the two vertices prior to the addition of the edge, but nonetheless this shows that such a region must be created by the operation.

So in all cases the quantity $V - E + F$ is unchanged, and the proof is complete. □

4 Two Graph-Theoretic Results

We now use Euler's formula to develop two graph-theoretic results. Our primary goal is to use these to prove the theorem that concludes this article, but these results still have independent interest.

The first result is a useful upper bound on the number of edges certain graphs can have.

Theorem 7. *If G is a simple connected finite planar graph with $V \geq 3$ vertices, then G has at most $3V - 6$ edges.*

Proof. Suppose G has E edges and consider a plane drawing of G having F faces. Let us count the number of ordered pairs (e, f) where e is an edge bounding the face f; suppose the resulting count is M. For a given edge e there are at most two faces that bound it, so $M \leq 2E$. But G is simple, so each face is bounded by at least three edges. Hence $M \geq 3F$. In this way we obtain the inequality $3F \leq 2E$.

Applying Euler's formula we find $3V - 3E + 3F = 6$. The inequality $3F \leq 2E$ implies $3F = 6 - 3V + 3E \leq 2E$ or, rearranging, $E \leq 3V - 6$. □

The second result is a crucial theorem guaranteeing the existence of at least one vertex of small degree in certain graphs.

Theorem 8. *If G is a simple connected finite planar graph with $V \geq 3$ vertices, then G has a vertex of degree at most five.*

Proof. Suppose G has E edges and the degrees of the V vertices are the V numbers $d_1, d_2, \ldots, d_V$. By use of the previous result we find

$$\sum_{i=1}^{V} d_i = 2E \leq 2(3V - 6) = 6V - 12.$$

This means that the arithmetic mean of the degrees of the vertices is at most $(6V - 12)/V = 6 - 12/V < 6$, so some vertex must have degree less than six. □

5 The Sylvester-Gallai Theorem

We are now in a position to provide a stunning proof of the following classical theorem of geometry:

Theorem 9. *Suppose there are $n \geq 3$ points in the plane, not all collinear. There must exist a line passing through precisely two of the points.*

Proof. Suppose that the points reside in the plane Π. Imagine placing the plane Π nearby (but not intersecting) the sphere S^2. For each of the n points in the plane there exists a line in space that intersects the given point and the center of S^2. In this way there is a one-to-one correspondence between points on the plane and pairs of antipodal points on the sphere. A line on the plane corresponds to a great circle on the sphere. We can now rephrase the assertion of the theorem as follows:

Suppose there are $n \geq 3$ pairs of antipodal points on the sphere S^2, not all on a single great circle. There must exist a great circle that contains precisely two of the antipodal pairs.

It is a principle of projective geometry that one can *dualize* the scene hence constructed. We replace a great circle by a pair of antipodal points and a pair of antipodal points by a great circle. We can set up this correspondence precisely as follows: to a great circle we associate the pair of points on the sphere that are furthest away from the plane containing the great circle, and to a pair of antipodal points we associate the great circle that lies in a plane orthogonal to the segment joining the points. With this dualization carried out, we obtain a third form of the assertion to be proved:

Suppose there are $n \geq 3$ great circles on the sphere S^2, not all of them passing through one point in common. There must exist a point that lies on precisely two of the great circles.

We recognize that the great circles create a simple connected finite planar graph drawn on the surface of the sphere: the vertices are the intersection points of the great circles, and the edges are the circular arcs of the great circles that connect the vertices. The graph is connected because any two great circles must intersect. It is planar by construction.

To show that it is simple, we imagine adding the $n \geq 3$ great circles onto the sphere one at a time. When there are only two circles present, it is possible that the graph is not simple, but the requirement $n \geq 3$ forces us to add another circle. Since not all of them may pass through a point in common, this new circle must cut the existing circles into edges, thereby creating a simple graph. In general, this can be repeated, and each time a new circle is added the new graph will be simple.

This means the previous theorem applies, and it guarantees the existence of a vertex of degree at most five. But the degree of any vertex will be even and at least four. Hence there is a vertex of degree four. This vertex must be the point of intersection of precisely two great circles. □

Sources: I referenced *Problem-Solving through Problems* by Loren C. Larson for the proof of Euler's formula, *Graph Theory* by Reinhard Diestel for finding precise graph-theoretic definitions, and *Proofs from the BOOK* by Martin Aigner and Gunter M. Ziegler for the proof of the Sylvester-Gallai theorem.