Periodic billiard orbits in convex polytopes

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Outline

1. 2 dimensions; the basic setup
2. Periodic billiards in 3 and 4 dimensions
3. Computer simulations
4. Further research

What is a billiard orbit?

Definition

Fix a polygon P in the plane. Ignoring friction, set into motion a point-mass m starting on one of the edges of P. The point-mass will travel along a straight trajectory except on collision with another edge of P, when it will be subjected to an elastic collision response (i.e., its velocity vector will be reflected in the line containing the edge). The overall trajectory traced out by m is its billiard orbit.

Assume the point-mass does not collide with the vertices.

Example

Pool. One can imagine playing pool not necessarily in a rectangle (the pool table) but in an arbitrary polygon P.

The basic question

Definition

A billiard orbit is periodic if the point-mass eventually returns to its starting position with the same velocity vector.

The basic question is this: in which polygons do periodic orbits exist?

Example

1. A square. Given a square S, the square whose vertices are the midpoints of the edges of S constitutes a periodic billiard trajectory.
2. An acute triangle. Given an acute triangle T, find the feet a, b, c of the three altitudes. The triangle with vertices a, b, c constitutes a periodic billiard trajectory sometimes called **Fagnano's orbit**.

Rational polygons

Definition

A rational polygon is a polygon every angle of which is a rational multiple of π.

Definition

A periodic billiard orbit in a polygon is **perpendicular** if it hits one of the edges at a right angle. The point-mass will thus hit this edge and then bounce directly backwards, following its trajectory in the opposite direction.

Theorem

Every rational polygon admits a perpendicular periodic billiard orbit.

UNFOLDING

Given a polygon P with edges labeled $e_1, e_2, \ldots, e_n$ and a sequence of integers $i_1, i_2, \ldots, i_m$, each of which is an element of $\{1, 2, \ldots, n\}$ such that no two adjacent sequence elements are equal, the basic tool is to unfold the polygon along this edge sequence.

1. Reflect the entire polygon P in the edge e_{i_1}. Maintain the labeling of edges in the reflected polygon P_2.
2. Reflect P_2 in its edge labeled e_{i_2}. Maintain the labeling of edges in the reflected polygon P_3.
3. ...
4. At the end, one will have a chain of copies of P.

The relation between unfolding and periodic orbits

1. Given a polygon P and its unfolded chain along some specified edge sequence, any periodic orbit in P will correspond to a straight line that stays entirely within the unfolded chain and connects a point on the starting edge of P with the corresponding image point on the final reflected polygon P_{m+1}.
2. This unfolding procedure generalizes to arbitrarily high dimensions.

Omnihedral billiards

Definition

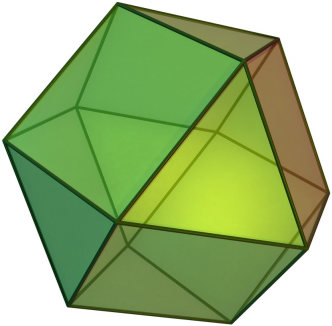
An omnihedral billiard is a periodic billiard that hits every face of an n-dimensional polytope exactly once.

3 and 4 dimensions

- Omnihedral periodic billiards have been found in all five regular polyhedra and in n-dimensional hypercubes and regular simplices.
- The 4-dimensional cross-polytope also admits such a billiard but for the 24-cell, 120-cell, and 600-cell it is not known whether any omnihedral periodic billiards exist.

The cuboctahedron

The cuboctahedron is a quasi-regular polyhedron with six square faces and eight triangular faces. The square faces share normals with the faces of a cube, and the triangular faces share normals with the faces of an octahedron.



The cuboctahedron

Periodic orbits

- Periodic orbits of orders 2, 4, and 6 that hit only square faces exist.
- It is not known whether an 8-periodic trajectory that hits all triangular faces exists.

Theorem

There exist no 3-periodic trajectories.

Given a vector v_i that hits a face with normal n_{i+1}, the reflected vector is given by

$$v_{i+1} = v_i - 2\langle v_i, n_{i+1}\rangle n_{i+1}.$$

The cuboctahedron

$$v_{i+1} = v_i - 2\langle v_i, n_{i+1}\rangle n_{i+1}. \qquad (1)$$

If a k-periodic trajectory has vectors $v_0, v_1, \ldots, v_{k-1}, v_k = v_0$, then using (1) recursively,

$$\sum_{i=0}^{k-1} \langle v_i, n_{i+1}\rangle n_{i+1} = 0.$$

Therefore a necessary condition for the existence of a k-periodic trajectory is that the normal vectors $n_0, \ldots, n_{k-1}$ corresponding to the faces hit by the trajectory must be linearly dependent.

The projection algorithm

Step 1 Start with a convex polyhedron P and a face sequence $f_1, f_2, \ldots, f_n$.

Step 2 Unfold the polyhedron P along the designated face sequence step by step.

Step 3 Let R_i be the reflection matrix for face f_i. Compute the product $R = R_1 R_2 \ldots R_n$ for an even number of reflections.

Step 4 If R is not the identity, let v be the real eigenvector with eigenvalue 1 (the axis of rotation). If R is the identity, let v be the translation of the origin after the unfolding process.

Step 5 Let Π be the plane through the origin orthogonal to v.

The projection algorithm, cont'd

Step 6 Before each reflection in the unfolding process, project the next face of reflection onto the plane Π, obtaining a polygon p_i in the plane. Compute the intersection I of the polygons $p_1, p_2, \ldots, p_n$.

Step 7 If R is the identity, then I will be the collection of starting points of periodic orbits that hit the faces $f_1, f_2, \ldots, f_n$. Otherwise I will be a set of candidate starting points, and further analysis by hand will be required.

The probabilistic algorithm

Step 1 Two points are randomly chosen on different faces of a convex polytope. The line joining them is the first segment of a billiard trajectory.

Step 2 The program extends the trajectory face by face. As soon as a face is encountered more than once, the program terminates.

Step 3 Steps 1 and 2 are repeated millions of times, and those billiards that hit each face exactly once and return to the original face are isolated.

Some results

- This algorithm was successful in isolating omnihedral face sequences for already solved polytopes: regular polygons, some regular polyhedra, and hypercubes.
- The algorithm has not found omnihedral face sequences for the remaining regular polychora.
- It did find a possible omnihedral face sequence in the cuboctahedron, but the projection algorithm reported that this sequence admits no periodic orbit.

A program for further research

- Classify periodic trajectories for regular polygons and polyhedra.
- Classify some of the trajectories in the unsolved cases.
- Run the algorithms over a long period of time to try to discover a periodic omnihedral orbit in the cuboctahedron and 24-cell.
- Determine whether or not there exists a periodic trajectory in the cuboctahedron that hits all the triangular faces (and no others).

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