Calculating linking numbers in two-fold branched covers of S^3

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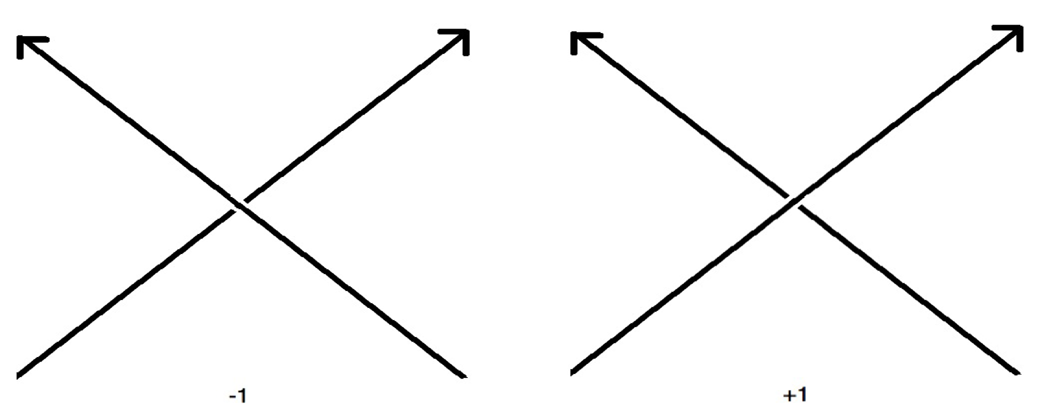
1. Abstract

A knot is an embedded one-dimensional submanifold of S^3. A link is a finite collection of disjoint knots that may loop around one another but do not intersect. Given a fixed knot J in S^3, we investigate those knots K that lift to a link L in the two-fold branched cover of S^3 branched over J, and we attempt to determine the linking of L. We place particular emphasis on the case where J is the unknot, which is geometrically realized as a circle in the three-sphere.

2. Linking Number

Let K_0 and K_1 be two oriented knots in S^3. We define their **linking number**, denoted $lk(K_0, K_1)$, as follows: choose a projection of the link in the plane and assign to each crossing ± 1 according to if the crossing is right-handed or left-handed. $lk(K_0, K_1)$ is the sum of the signs of the crossing. While the projection is not an invariant of the link, it can be shown using Reidemeister moves that the linking number is.



3. Branched Covers

A **branched cover** is a continuous map $p : X \to M$ between compact manifolds such that for a submanifold $A \subset M$, $M - A$ is exactly the collection of points that are evenly covered by p. We call A the branch set of p. For example, the map

$$z \to \frac{z^k}{|z|^{k-1}}$$

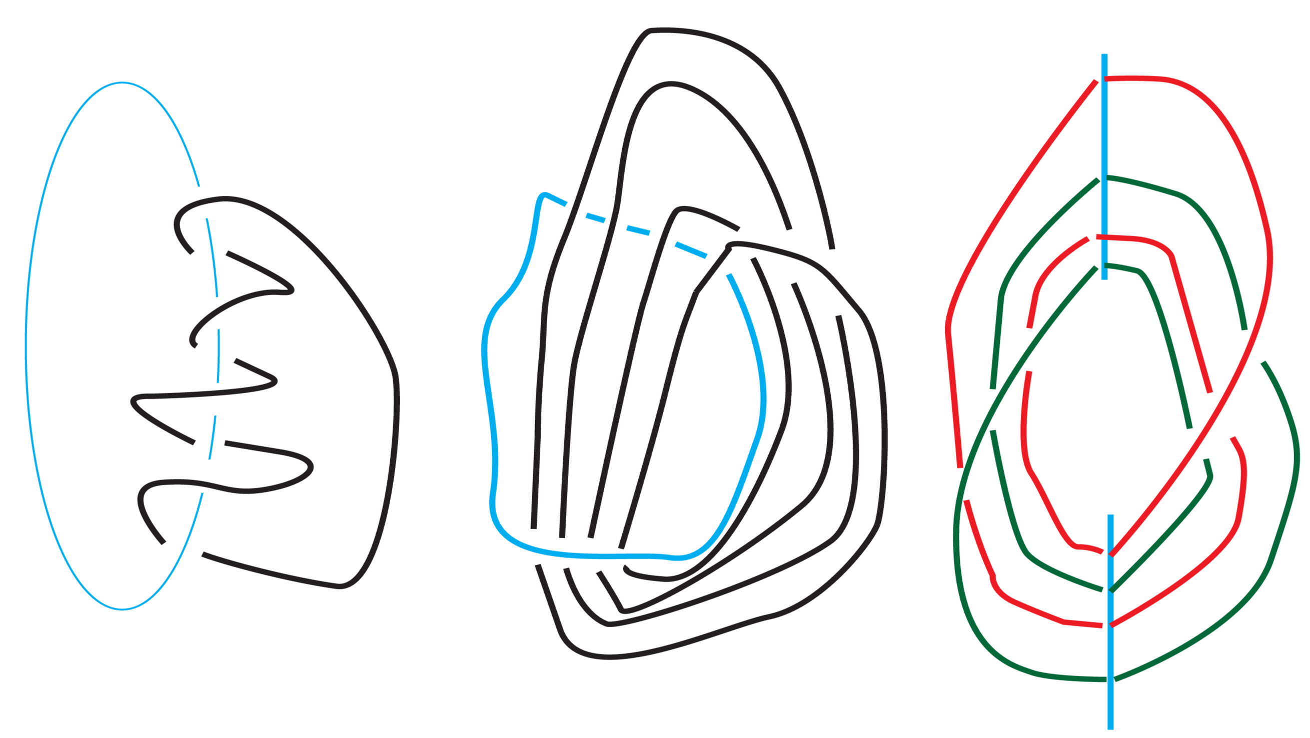
is a branched cover of the unit disk.

4. Project Summary

Let J be a fixed knot in S^3. If K is a knot in the exterior of S^3 that lifts to the 2-fold branched cover of S^3, branched over J, then its lift is a link of two components, K_0 and K_1. The goal of this project is to determine the linking number $lk(K_0, K_1)$. We take J to be the unknot, in which case the 2-fold branched cover is S^3.

5. Method

Let K be a knot in the exterior of the unknot J. We say that K is in **standard position** if in the projection of $K \amalg J$ to the plane, all crossings where J passes over K are adjacent and all those where J passes under K are adjacent. We assume that K is in standard position, as any knot is isotopic to a knot in standard position. The lift of K to the 2-fold branched cover of J lies outside the pre-image of the branch set. Hence, it is sufficient to lift K to $L = K_0 \amalg K_1$ in the 2-fold regular cover of the exterior of J and calculate $lk(K_0, K_1)$ there. Let K' be the tangle obtained by "cutting" the strands of K as they pass through the disk bounded by J. We obtain a projection of L in the plane by "gluing" two copies of K' together in an orientation-preserving way.



From left to right: a knot K in S^3, K in general position, and the lift of K to the 2-fold branched cover of the unknot.

6. Results

The following theorem is proved using the method outlined in Box 5 and a counting argument.

Theorem. *Let K be a knot that links the unknot exactly $2n$ times, in the same direction, and does not cross itself. Then $lk(K_0, K_1) = n$.*

The case when $n = 4$ is depicted in the figure below.

7. Applications

Knot theory has significant applications in the real world especially biology. DNA is constantly knotted and unknotted, and these processes are vital to the survival of the cell. Until recent developments in knot theory, experts had a difficult time analyzing the ways DNA knots and unknots. Abstract mathematical theory has answered many questions about these biological processes. Knot theory enters into chemistry in determining chirality of molecules. In physics, knot theory is used in the study of quantum gravity (an area designed to branch the gap between quantum mechanics and general relativity). Finally, knot theory has applications to other areas of mathematics, such as the study of 4-manifolds.

7. References

References

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